# Analytically solvable model of charge dispersion in nuclear fission

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Using the asymmetric two-center shell model potentials in parametrized forms, the time-dependent Schrödinger equation is solved analytically for the charge dispersion yields. We obtain a Gaussian function, characterized by a most probable charge and the width of the distribution. The hypothesis of an unchanged charge density and a minimum potential energy are given as limiting cases. Calculations are made for $^{236}$U and $^{252}$Cf.

NUCLEAR REACTIONS Nuclear fission, fragmentation theory, two-center shell model, time-dependent Schrödinger equation, analytical solution, Gaussian charge distribution, most-probable charge, unchanged charge density and minimum potential energy postulates, $^{236}$U, $^{252}$Cf.

Empirically, the independent fractional charge dispersion yields, for a given mass chain, have been represented by a Gaussian distribution[1]

$$P(z)=(c\pi)^{-1/2}\exp[-(z-z_p)^2/c] \tag{1}$$

characterized by the "most-probable charge" $z_p$ (the mean value) and the width of the distribution $c$. Also, many empirical postulates were proposed for the charge divisions in fission; namely, the unchanged charge distribution (UCD), the equal charge displacement (ECD), and the minimum potential energy (MPE) postulates. Recently, however, a complete theory without any free parameter has been given by one of the authors and collaborators,[2] where the charge dispersion yield is calculated by numerically solving a stationary Schrödinger equation

$$H\psi_{R\xi}^{(\nu)}(\xi_z)=E_{R\xi}^{(\nu)}\psi_{R\xi}^{(\nu)}(\xi_z) \tag{2}$$

with the Hamiltonian

$$H=-\frac{\hbar^2}{2(B_{\xi_z\xi_z})^{1/2}}\frac{\partial}{\partial\xi_z}\frac{1}{(B_{\xi_z\xi_z})^{1/2}}\frac{\partial}{\partial\xi_z}+V(\xi_z)_{R\xi} \tag{3}$$

in a dynamical variable of charge asymmetry,

$$\xi_z=(z_H-z_L)/(z_H+z_L),$$

between heavy ($H$) and light ($L$) fragments. The coordinate of relative separation $R$ of the two fragments and the mass symmetry

$$\xi=(A_H-A_L)/(A_H+A_L)$$

of fragments are kept fixed, thereby assuming decoupling of these parameters with $\xi_z$. The potential $V(\xi_z)$ at fixed $R$ and fixed $\xi$ is obtained in the Strutinsky manner from the asymmetric two-center shell model (ATCSM) using the adiabatic approximation for the shape parameters. The mass parameters $B_{ij}$ are consistently obtained from the cranking formula in the Bardeen-Cooper-Schrieffer (BCS) formalism. The theory gives the charge distributions of the Gaussian type that are independent of the nuclear temperatures and fit the experimental data both in peak ($z_p$) and width ($c$). Within the general framework of this theory, in this paper we propose a simplified model to solve a time-dependent Schrödinger equation analytically for $|\psi(\xi_z,t)|^2$, which explicitly gives a Gaussian functional form for the fractional charge dispersion yields. The resulting Gaussian function fits the experiments nicely, and the UCD and MPE hypotheses are incorporated in the expression for $z_p$.

The time-dependent Schrödinger equation in $\xi_z$, for fixed $\xi$, is given by

$$H\psi_\xi(\xi_z,t)=i\hbar\frac{\partial}{\partial t}\psi_\xi(\xi_z,t), \tag{4}$$

where the charge dispersion potential $V(\xi_z)$ for fixed $R$ and fixed $\xi$ calculated using the ATCSM can always be approximated nicely by a harmonic oscillator potential

$$V(\xi_z)=\tfrac{1}{2}k(\xi_z-\xi_z^{\min})^2. \tag{5}$$

Here $\xi_z^{\min}$ refers to the minimum in potential energy surface $V(\xi_z)$, and $k$ is the force constant.

We solve Eq. (4) analytically, under the initial condition of a very narrow Gaussian distribution,

 

$$\psi(\xi_z,t=0)=\exp[-\tfrac{1}{2}(\xi_z-\xi_z^{\rm UCD})^2/\Gamma_i]\Bigg/\left[\int_{-\infty}^{\infty}\exp\{-(\xi_z-\xi_z^{\rm UCD})^2/\Gamma_i\}(\bar{B}_{\xi_z})^{1/2}d\xi_z\right]^{1/2}, \quad (6a)$$

that gives the deviations from the UCD charge asymmetry, $\xi_z^{\rm UCD}$. $\bar{B}_{\xi_z}$ is the cranking mass $B_{\xi_z\xi_z}$ averaged over the $\xi_z$ coordinate, and $\Gamma_i$ gives the initial width. Also, the wave function $\psi(\xi_z,t)$ is expanded in terms of the stationary harmonic oscillator wave functions $\varphi_n$,

$$\psi(\xi_z,t)=\sum_{n=0}^{\infty}a_n\varphi_n(\xi_z)\exp[-i(n+1/2)\omega t], \quad (7)$$

with

$$a_n=\int_{-\infty}^{\infty}\varphi_n^*(\xi_z)\psi(\xi_z,0)d\xi_z \quad (8)$$

and

$$\omega=(k/\bar{B}_{\xi_z})^{1/2}. \quad (9)$$

It may be noted here that the wave function is normalized in the interval $-\infty$ to $\infty$, though the range of definition of our physical coordinate is $-1\le\xi_z\le 1$. This is possible since in our model the initial width $\Gamma_i$ is very small, such that

$$\int_{-1}^{1}\exp(-x^2/\Gamma_i)dx=\sqrt{\Gamma_i}\int_{-1/\sqrt{\Gamma_i}}^{1/\sqrt{\Gamma_i}}\exp(-y^2)dy \underset{\substack{\lim\\ \text{small }\Gamma_i}}{\approx} \sqrt{\Gamma_i}\int_{-\infty}^{\infty}\exp(-y^2)dy=\int_{-\infty}^{\infty}\exp(-x^2/\Gamma_i)dx, \quad (6b)$$

where $x=\xi_z-\xi_z^{\rm UCD}$.

We obtain

$$|\psi(\xi_z,t)|^2=(\bar{B}_{\xi_z}\pi\Gamma(t))^{-1/2}\exp[-(\xi_z-\xi_{z_p})^2/\Gamma(t)], \quad (10)$$

which is a Gaussian function with the half-width, in terms of

$$\beta=(\Gamma_i\omega\bar{B}_{\xi_z}/\hbar)^{1/2},$$

of

$$\Gamma(t)=\Gamma_i(1+\beta^4-(1-\beta^4)\cos2\omega t)/2\beta^4, \quad (11)$$

and the mean value

$$\xi_{z_p}=\xi_z^{\min}-(\xi_z^{\min}-\xi_z^{\rm UCD})\cos\omega t. \quad (12)$$

From Eq. (12) we get, for both the heavy and light fragments, the explicit expressions for the most probable charges:

$$Z_p=\tfrac{1}{2}z[1\pm(\xi_z^{\min}-(\xi_z^{\min}-\xi_z^{\rm UCD})\cos\omega t)]. \quad (13)$$

Equations (10)–(13) present very interesting results. Equation (10) gives the probability, as a function of time, of the Gaussian form that is used empirically in Eq. (1). Equation (11) shows that the half-width $\Gamma(t)$ oscillates periodically with frequency $2\omega$ between the maximum and minimum values $\Gamma_i$ and $\Gamma_i/\beta^4$, respectively, at the times

$$t=\frac{\pi}{\omega}(n+\tfrac{1}{2})$$

and $(\pi/\omega)n$; $n=0,1,2,3,\ldots$. Equations (12) and (13) give, respectively, the mean values and the most probable charges that are shown to oscillate periodically but with frequency $\omega$ and are independent of the initial width $\Gamma_i$. Furthermore, Eq. (13) shows that $z_p=z^{\rm UCD}$ in the limit of $\cos\omega t=1$ and equals the MPE value $z^{\min}$, in another limit of $\cos\omega t=0$. Also, $z_p\approx z^{\min}$ for the limit $\cos\omega t=-1$ provided the difference $(\xi_z^{\min}-\xi_z^{\rm UCD})$ is very small. Since $\omega$ is fixed for a given system [Eq. (9)], the determination of both $\Gamma$ and $z_p$, and hence of probability, apparently depends only on the proper estimation of the scission time $T$.

We estimate the scission time $T$ by considering the relative motion of the fissioning system from the top of the barrier to the scission point ($R_{\rm sc}$). The barrier, represented by height $V_B$ and position $R_B$, can be approximated by a quadratic expression

$$V(R)=V_B-\tfrac{1}{2}K(R-R_B)^2,\ \ R>R_B. \quad (14)$$

Newton's equation of motion for such a potential (force $=-\partial V/\partial R$) is given by

$$\mu \ddot{R}(t)=K[R(t)-R_B]. \quad (15)$$

The reduced mass $\mu$ is used with a view that the relative motion is decoupled or, at least, weakly coupled[2] to the charge transfer $\xi_z$, since we find that in general $\mu \approx \bar{B}_{RR}$ (the cranking mass averaged in relative coordinate) and $B_{R\xi_z} \ll (B_{RR}B_{\xi_z\xi_z})^{1/2}$. Solving Eq. (15), under the initial conditions of

$$R=R_i \text{ at } t=0 \text{ and } R=R_{sc} \text{ at } t=T, \quad (16)$$

where $R_i$ is the initial position, located just pass the saddle, we obtain

$$T=\frac{1}{\omega_r}\ln[(R_{sc}-R_B)/(R_i-R_B)], \quad (17)$$

with

$$\omega_r=(K/\mu)^{1/2}. \quad (18)$$

We have used here the condition of conservation of energy at the top of the barrier, i.e.,

$$V_B=\tfrac{1}{2}\mu\dot{R}^2(t)+V(R). \quad (19)$$

This equation, with Eq. (14), gives the initial velocity

$$v_i=\dot{R}(t=0)=\left[\frac{2}{\mu}(V_B-V(R_i))\right]^{1/2}$$
$$=\omega_r(R_i-R_B). \quad (20)$$

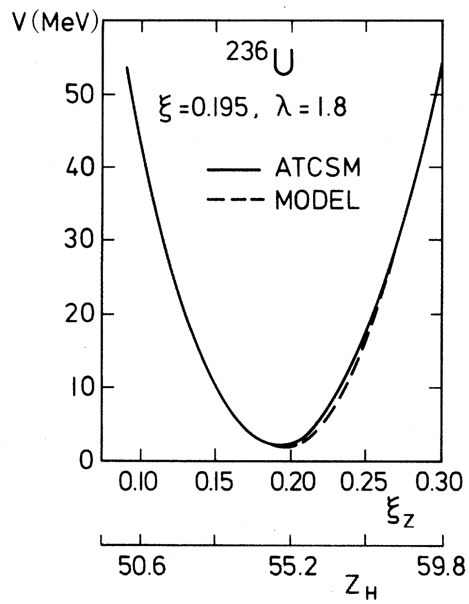


FIG. 1. Charge dispersion potential for $^{236}$U.

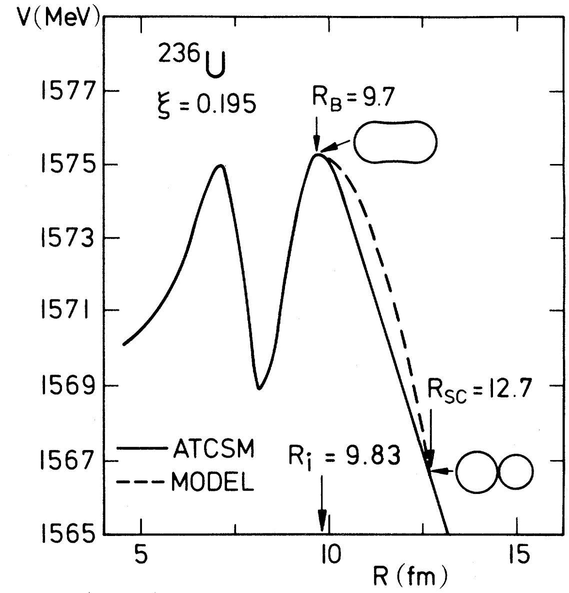


FIG. 2. Scattering potential and the saddle and scission nuclear shapes for $^{236}$U.

The choice of $R_i > R_B$ is made in view of the assumed finite, small width of the initial distribution (6) (see also Fig. 4). As $R_i$ increases, Eq. (17) shows that $T$ decreases.

The final charge distribution yield at the scission time $T$ is the probability $|\psi(\xi_z,T)|^2$ scaled to the fractional charge yield for the charge, say $z_H$, in the interval $d\xi_z(=2/z)$:

$$P(z_H)=\frac{2}{z}\left[\frac{1}{\pi\Gamma(T)}\right]^{1/2}\exp\left[-\frac{(\xi_z-\xi_{z_p})^2}{\Gamma(T)}\right]. \quad (21)$$

This gives us a Gaussian distribution function which differs from Eq. (1) only in that the two Gaussians have different amplitudes.

We have tested our model by calculating the charge distributions in the fission of $^{236}$U and $^{252}$Cf for the mass asymmetries $|\xi|=0.195$ and 0.090, respectively, which for the spontaneous fission represent the mass chains $A_H=141$, $A_L=95$ and $A_H=137.4$, $A_L=114.6$. However, the calculations are compared with the experimental data[3] for the heavy mass chains only.

Figures 1 and 2 give, respectively, the charge dispersion potential $V(\xi_z)$ and the scattering potential $V(R)$ for the illustrative case of $^{236}$U, calculated by using the ATCSM and the model expressions. The model potentials are shown to represent the exact ATCSM potentials very satisfactorily. The values obtained for the constants are $k=9450.52$ MeV, $K=1.91$ MeV fm$^{-2}$.

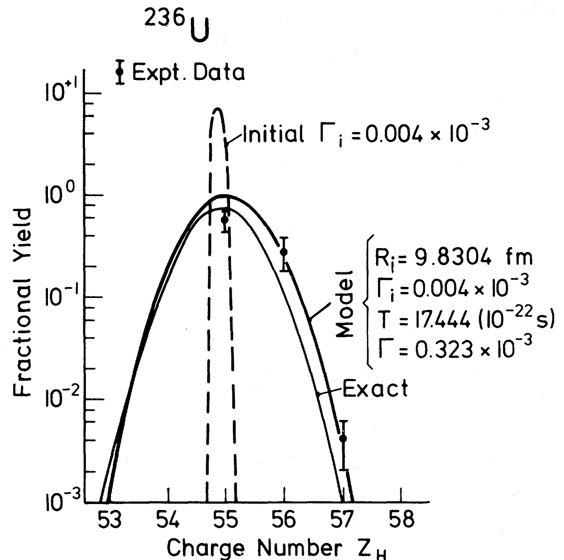


FIG. 3. Theoretical charge distribution yields for $^{236}U$ compared with the experimental data of Ref. 3 for $A=141$ and 142.

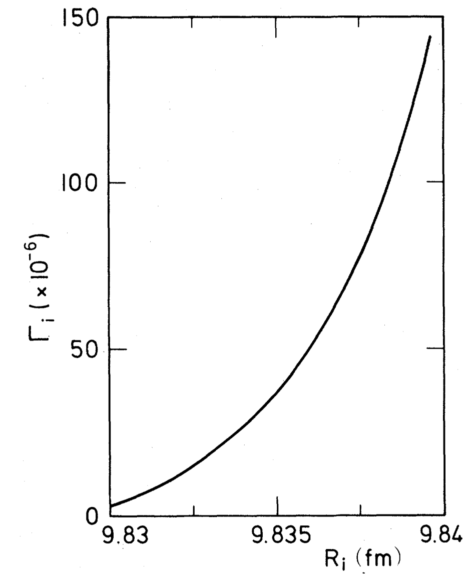


FIG. 4. Variation of initial width $\Gamma_i$ with the initial position $R_i$ for the calculated final charge distribution yield shown in Fig. 3. Notice the $R_i$ scale.

Using the corresponding positions of $R_B$, $R_{sc}$, and $R_i$, as indicated in Fig. 2, and the reduced mass $\mu=56.76$ ( in units of proton mass $M_p$) for $^{236}U$, we obtain from Eq. (17) the scission time $T=17.44\times10^{-22}$ sec. For $^{252}Cf$ we get $T=21.04\times10^{-22}$ sec. The two values are typical of the adiabatic fission process.[1]

The calculated fractional charge distribution yield is plotted in Fig. 3 for the model [Eq. (21)] and compared with the experimental data[3] as well as with the exact theoretical calculations of Ref. 2. The value of $\bar{B}_{\xi_z}=1.77\times10^4 M_p$ fm$^2$. The model is shown to fit the data nicely and is also comparable with the exact theoretical calculations, except for the peak height. However, the rise in the peak height is already known[2] to be an effect of the use of the averaged mass parameter $\bar{B}_{\xi_z}$. The values of $Z_p$ for the model, the exact calculations, and the UCD and MPE hypothesis are, respectively, 54.97, 55.10, and 54.95 and 54.97. There is hardly any significant difference in the various predictions. Calculations for $^{252}Cf$, $\xi=0.090$, give the same result with $z_p=53.44\pm0.02$.

We have also tested the accuracy of the predictions of our model. For an equally good fit to the data, the variation of $\Gamma_i$ and $R_i$ is shown in Fig. 4. It is interesting to observe that in spite of our treating the $R$ and $\xi_z$ motions separately, the fixing of their corresponding parameters $R_i$ and $\Gamma_i$ in our model depend on each other. The width $\Gamma_i$ of the initial distribution increases with the increase in the value of $R_i$, as one would expect. Our results for $^{252}Cf$ are identical.

The time evolution of the process (the widths, the mean values, and the probability) can also be studied in our model by using the explicit expressions (10)–(13) or by solving the time-dependent Schrödinger equation (4) numerically at different times from 0 to $T$. So far, this has been of more interest for the heavy ion collisions. It has been observed experimentally[4] that in collisions like 8.3 MeV/nucleon $^{56}Fe$ on $^{56}Fe$, $^{165}Ho$, $^{209}Bi$, and $^{238}U$ or 430 MeV $^{86}Kr$ on $^{92,98}Mo$, the charge widths at constant mass asymmetry saturate after an initial rise, and the mean values show a consistent increase with the energy loss (or reaction time). This fast charge equilibration process has been treated both quantum mechanically[5] (harmonic oscillator coupled to a thermal bath) and statistically[6] (transport theory) with almost equal success. Within a single harmonic oscillator model, similar to ours, it has been shown[7] that a dissipative mechanism (like friction) is a must for the damping of the periodic oscillations given by this model for the widths and mean values. For the fission of natural nuclei, however, there are no systematic measurements for the heavy mass products, though some data are available[8] for the light mass chains ($A=98-104$). In any case, for a dissipative phenomena, a mechanism for losing energy has to be included in a theory. This may mean invoking frictional forces or other collective and intrinsic degrees of freedom in our model.

In conclusion, we have shown that a simple analytical solution of the time-dependent Schrödinger equation for the parametrized charge dispersion and scattering potentials of the fragmentation theory give an explicit Gaussian functional form for the charge distribution yield with the most probable charge containing the UCD and MPE hypothesis as limiting cases. For the examples of $^{236}U$ and $^{252}Cf$ studied here, both the hypothesis of UCD and MPE are found to be equally satisfactory.

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